An efficient integration technique for the voxel-based Finite Cell Method

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Abstract

The Finite Cell Method (FCM) is a fictitious domain approach based on hierarchical Ansatz spaces of higher order. The method avoids time-consuming and often error-prone mesh-generation and favorably exploits Cartesian grids to embed structures of complex geometry in a simple shaped computational domain thus shifting parts of the computational effort from mesh generation to the computation within the embedding finite cells of regular shape. This paper presents an effective integration approach for voxel-based models of linear elasticity that drastically reduces the computational effort on cell level. The applied strategy allows the pre-computation of an essential part of the cell matrices and vectors of higher order, representing stiffness and load, respectively. Several benchmark problems show the potential of the proposed method in particular for heterogeneous material properties as common in biomedical applications based on computer tomography scans. The applied strategy ensures a fast computation for time-critical simulations and even allows user-interactive simulations for models of moderate size at a high level of accuracy.

Keywords: fictitious domain method; Finite Cell Method; voxel models; high-order FEM; solid mechanics

1 Introduction

Voxel models generated by recursive bisection [13, 36] or derived from x-ray images and quantitative computer tomography scans (qCT-scan), respectively, are often used to overcome a time-consuming and error-prone mesh-generation for the numerical simulation of structures of high geometric complexity. Such simulations include e.g. seismic analyses and ground water flow in soil mechanics [16, 6], shape and topology optimization of complicated structures [9, 21, 1, 33], stability analyses of foam structures [25] and numerous examples from biomedical fields as bone [24, 39, 40, 23, 17] or dental mechanics [18, 14]. For Finite Element schemes the voxel approach avoids fine granular meshes but still is often limited in terms of accuracy due to a constant element-wise material assignment. In particular in terms of computational efficiency it often requires means of parallelization to generate solutions with a reasonable time effort [34, 19, 8].

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The following contribution introduces an extension of the Finite Cell Method [20, 10], a high-order fictitious domain method that favorably applies the simple nature of Cartesian grids and turns out to be highly suited for time-critical voxel-based simulations of complex structures in various fields of numerical simulation. The Finite Cell Method recently has been successfully applied for geometrically nonlinear analyses of foam-like structures with various refinement strategies [26, 25], advection-diffusion problems [6] and applications in the field of bone mechanics [24] as well as to the numerical homogenization of heterogeneous and foamed materials [28, 11]. With the extension of a pre-computation scheme the method even is capable to produce sufficient update rates for user-interactive simulations in patient-specific femur analyses [38]. The developments described in the following focus on a time-efficient extension of the FCM for linear elasticity problems that partly can be applied also for the linearized equations of non-linear problems. However the major effort of such problems remains computationally expensive and cannot directly profit from the presented concepts. Beside the algorithmic reduction of the time complexity the performance of the implementation profits from a multi-core parallelization with expected parallel efficiency on eight cores. Owing to the exploited Ansatz space of higher order [32, 31], the method provides properties comparable to the p-FEM with regard to convergence behavior and error control. A flexible sub-cell integration scheme proposed in [10] allows to accurately capture multi-material interfaces and heterogeneous material distributions as common in biomedical problems, a scheme that serves as basis for the proposed extension in this contribution.

This paper is organized as follows: Section 2 provides a brief overview about the basic idea of the Finite Cell Method. Section 3 introduces the fast integration concept for three-dimensional problems with a special focus on qCT-derived voxel data. Section 4 provides three numerical examples to illustrate the performance of the proposed method. The paper is summarized with the conclusions in Section 5.

2 The Finite Cell Method

The Finite Cell Method [20, 10] is a fictitious domain method based on high-order polynomial Ansatz spaces that embeds the physical domain $\Omega$ of arbitrary complex geometry in an extended, fictitious domain $\Omega_{FD}$ of significantly simpler shape (Fig. 1). The method favorably exploits the properties of Cartesian grids and thus avoids numerically expensive mesh generation.

The grid structure is applied to decompose the embedding domain $\Omega_{FD}$ into a set of finite cells $\Omega_C$ covering the physical domain $\Omega$. Cells that are completely outside $\Omega$ are discarded to reduce the effort of computation and to support numerical stability of the formulation. In general the cell boundary $\partial \Omega_C$ is assumed traction free. Homogeneous and inhomogeneous Dirichlet and Neumann boundary conditions of the physical domain are directly applied to the boundary $\Gamma_D \cup \Gamma_N = \Gamma$. Surface traction on the non-homogeneous Neumann boundary $\Gamma_N$ is applied by integration over a parametrized surface patch on $\Gamma$ that can be obtained with negligible effort e.g. from a locally confined triangulation. In [10] the application of Neumann boundary conditions on the embedded structure is discussed in detail.

Various studies show that a weak enforcement of Dirichlet boundary conditions provide a reliable strategy to prescribe primals at arbitrary positions along $\Gamma_D$. So far the penalty method [35] and a Nitsche type weak enforcement [24] of the essential boundary conditions have been tested successfully.

With penalization of the stress tensor $\mathbf{\sigma}(\alpha) = \alpha \mathbf{\sigma}$ in the fictitious domain the formulation ensures consistency with the original problem. The function $\alpha(x)$ indicates the true structure $\Omega$ within the fictitious domain of the cell structure

$$
\alpha(x) = \begin{cases} 
\alpha = 1 & \forall x \in \Omega \\
\alpha = \epsilon & \forall x \in \Omega_C \setminus \Omega 
\end{cases}
$$

with $\epsilon$ chosen small enough to account for the physical domain only. Typical values for $\epsilon$ vary in the range between $10^{-4}$ and $10^{-14}$ and can be found e.g. in dependence of the stiffness of the structure to ensure both, accuracy and numerical stability [24].

With (1) the weak formulation of the linear elasticity problem for the extended problem is specified according to the principle of virtual work

$$
\delta W_\sigma(u, \delta u) = \int_{\Omega_C} \delta \varepsilon : \mathbf{\sigma}(\alpha) \, d\Omega - \int_{\Omega_C} \delta u \cdot \mathbf{p}(\alpha) \, d\Omega - \int_{\Gamma_N} \delta u \cdot \mathbf{t}_0 \, d\Gamma = 0
$$

including the displacement vector $u$, linear strains $\varepsilon$, the virtual quantities $\delta u$ and $\delta \varepsilon$ and the volume load vector $\mathbf{p}$. The volume loads in the fictitious part of $\Omega_C$ are penalized in analogy to the stresses with $\alpha$.

The Finite Cells are implemented as hexahedral elements according to the principles of tensor product elements of the Finite Element Method. The unknown displacement field $u(x)$ is approximated with hierarchical piecewise defined polynomials of higher order $N_i(\xi, \eta, \zeta)$ specified.

in the standard hexahedral \((-1 \leq \xi, \eta, \zeta \leq 1)\) (see [30])

\[
\mathbf{u} = \sum_a N_a(\xi, \eta, \zeta) \mathbf{U}_a
\]

(3)

\[
\delta \mathbf{u} = \sum_a N_a(\xi, \eta, \zeta) \delta \mathbf{U}_a
\]

(4)

with \(\mathbf{U}_a\) and \(\delta \mathbf{U}_a\) denoting the unknown degrees of freedom. The approximation of the linear strain tensor \(\epsilon\) and corresponding virtual quantity applies the standard strain operator \(\mathbf{B}(\xi, \eta, \zeta)\) that is obtained from differentiation of (3) with respect to the global coordinates \((x, y, z)\), applying the chain rule.

Following the Bubnov-Galerkin approach equations (3) and (4) are substituted into the weak form (2) providing a discrete finite cell formulation

\[
K \mathbf{U}_a = \mathbf{P}
\]

(5)

with the \((N \times N)\)-matrix \(K\) representing the system stiffness and corresponding system load vector \(\mathbf{P}\).

3 Cell integration scheme for voxel data

Rapid and simple grid generation is one of the major advantages of the Finite Cell Method as compared with tedious and often error-prone mesh generation for complex domain geometries or multi-material interfaces. With the indicator function \(\alpha(x)\), specified in the previous section, that decides if points are inside or outside the physical domain, an essential part of the modeling effort is shifted to the governing integrals on cell level. The integrals (6) and (7) are computed numerically with Gauss quadrature. In the general case of non-voxelized solid structures an adaptive integration scheme is necessary to accurately capture the true boundary of the physical domain and to confine the integration error and the numerical effort [26]. In [10] a sub-cell strategy is introduced that allows to arbitrarily densify the quadrature points within \(\Omega_{\text{CT}}\) thus supporting adaptive integration on various granularity levels. In combination with a locally restricted octree generation that turns out to be an efficient and fast voxelization approach, the true domain is accurately identified (see e.g. [26, 25, 10]).

The shape of voxelized domains is directly given either by the assembly of voxels or by the value of each voxel in the case of qCT-derived voxel data. For both cases a regular sub-cell scheme is applied, decomposing each finite cell into \((m_x \times m_y \times m_z)\) sub-cells for integration.

The number of quadrature points required to exactly integrate polynomials depends on the polynomial degree. Despite the polynomial character of the integral (6) that has a constant and diagonal Jacobian \(J\), \((p + 1)^3\) quadrature points are in general applied for integration of cells with shape functions of degree \(p\). Accepting an approximation error of negligible size, the sub-cell approach allows a reduction of the number of quadrature points for each sub-cell and even more it ensures a regular distribution to best possibly capture a discrete material distribution of heterogeneous material properties.
3. CELL INTEGRATION SCHEME FOR VOXEL DATA

3.1 Pre-computation on cell level

With the quantities \( u, \delta u \) and \( \varepsilon \) defined in the previous section, the stiffness matrix of cell \( c \) can be obtained resulting in

\[
K_c = \int_{\xi} \int_{\eta} \int_{\zeta} B^T \alpha C B \det J \, d\zeta \, d\eta \, d\xi
\]

with a constant and diagonal Jacobi matrix \( J \) of the cell geometry mapping and the elasticity matrix \( C \). Similarly the cell volume load vector is given as

\[
P_c = \int_{\xi} \int_{\eta} \int_{\zeta} N^T \alpha p \det J \, d\zeta \, d\eta \, d\xi
\]

For a pre-computation of the integrals (6) and (7) the following strategy is applied: The material properties of the \((n_x \times n_y \times n_z)\) voxels per finite cell are assumed constant for each voxel. This is especially true for qCT-derived voxel data and is favorably exploited to model a discrete and heterogeneous material distribution. With this property the integrals (6) and (7) representing cell stiffness and cell load, respectively, are replaced by the sum over \((n_x \times n_y \times n_z)\)-integrals, each defined over the domain of a single voxel:

\[
\hat{K}_c = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} K_{ijk}
\]

\[
\hat{P}_c = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} P_{ijk}
\]

Material properties and load intensity as the essential and voxel-wise changing quantities are uniquely identified by the indices \( i, j, k \) that are used to define the integration limits for each voxel integral, according to the normalized coordinate directions \( \xi, \eta, \zeta \)

\[
t_{\xi,i} = -1.0 + 2 \frac{i-1}{n_x}, \quad i = 1, \ldots, n_x
\]

\[
t_{\eta,j} = -1.0 + 2 \frac{j-1}{n_y}, \quad j = 1, \ldots, n_y
\]

\[
t_{\zeta,k} = -1.0 + 2 \frac{k-1}{n_z}, \quad k = 1, \ldots, n_z
\]

The voxel integrals then follow as

\[
K_{ijk} = \int_{t_{\xi,i}}^{t_{\xi,i+1}} \int_{t_{\eta,j}}^{t_{\eta,j+1}} \int_{t_{\zeta,k}}^{t_{\zeta,k+1}} B^T \alpha C_{ijk} B \det J \, d\zeta \, d\eta \, d\xi
\]

\[
P_{ijk} = \int_{t_{\xi,i}}^{t_{\xi,i+1}} \int_{t_{\eta,j}}^{t_{\eta,j+1}} \int_{t_{\zeta,k}}^{t_{\zeta,k+1}} N^T \alpha p_{ijk} \det J \, d\zeta \, d\eta \, d\xi
\]

In the special case of isotropic material properties, the elasticity matrix \( C_{ijk} \) for each voxel can be further split into two independent parts according to the Lamé constants \( \lambda_{ijk} \) and \( \mu_{ijk} \) (see [4])

\[
C_{ijk} = \lambda_{ijk} C^\lambda + \mu_{ijk} C^\mu
\]
where

\[
C^\lambda = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

and

\[
C^\mu = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

are constant matrices (Eq. (13)) and can therefore be written as

\[
\hat{K}_c = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} (\lambda_{ijk} K^\lambda_{ijk} + \mu_{ijk} K^\mu_{ijk}).
\]

It is worth to note that the new defined quantities $K^\lambda_{ijk}$, $K^\mu_{ijk}$ are independent of the voxel-wise material properties and can be pre-computed in dependence of the polynomial degree of the Ansatz, the number of voxels per cell ($n_x \times n_y \times n_z$) and the voxel spacing ($s_x, s_y, s_z$). Thus the impact of changing material properties during the simulation of structures reduces to a modification of the Lamé constants that simply scale the pre-computed voxel stiffness, followed by summation over all stiffness contributions. Contributions from the extension domain are penalized with $\alpha$ as defined in (1).

For qCT-derived voxel models the spacing of the voxels depends on the spatial resolution of the CT-scanner. Assuming the same voxel spacing $s_x = s_y$ in the $x - y$-plane of each qCT-slice the dependence of the quantities $K^\lambda_{ijk}$ and $K^\mu_{ijk}$ reduces from the three parameters $s_x, s_y, s_z$ to two parameters $r, s_z$.

\[
K^\lambda_{ijk} = s_z (K^\lambda_{ijk}^0 + r K^\lambda_{ijk}^1 + r^2 K^\lambda_{ijk}^2)
\]

\[
K^\mu_{ijk} = s_z (K^\mu_{ijk}^0 + r K^\mu_{ijk}^1 + r^2 K^\mu_{ijk}^2)
\]

with $r := s_x/s_z$, thus valid for any type of qCT-scan [37].

The voxel integrals are solved numerically applying a $p + 1$ point Gauss-integration. For many applications the pre-computation scheme for the fine ($n_x \times n_y \times n_z$) voxel resolution can be replaced by a coarser resolution of ($m_x \times m_y \times m_z$) sub-cells, each representing a chunk of ($n_x/m_x \times n_y/m_y \times n_z/m_z$) voxels with averaged properties when indicated.

Figure 2 provides an overview about the storage requirements of the pre-computed matrices of the present implementation that exploits the hierarchic shape functions of the p-version trunk space [32, 30].

### 3.2 Efficiency analysis

The following academic example gives an impression about the efficiency of the pre-computation approach for time-critical computations such as e.g. user-interactive simulations [38]. The computational model consists of 1000 finite cells each covering $10 \times 10 \times 10$ voxel. The example
4. NUMERICAL EXAMPLES

The following examples were selected to verify the performance of the proposed method in terms of accuracy and numerical efficiency. All computations were performed with the p-FEM-analysis platform AdhoC [12] that has been extended by an implementation of the pre-computation concept. All examples focus on reduction of the numerical effort for the computation of the cell-stiffness. The granularity level for all computations is either set to \((5 \times 5 \times 5)\) or \((10 \times 10 \times 10)\) voxel per finite cell.

Figure 2: Memory allocation of the precomputed matrix \(K^{\lambda_0}\) for increasing polynomial degree with fixed resolution of \(10 \times 10 \times 10\) voxel per finite cell. For the complete pre-computed stiffness matrix \(K_c\) the shown values increase by a factor 6.

Table 1 gives an overview about the computation times for the competing integration schemes, revealing an acceleration factor that scales weakly superlinear with the number of degrees of freedom. It has to be mentioned that for this comparison and for the comparisons of all following examples the measurements for the pre-computation scheme assume that the set of pre-computed matrices is already available. With the scheme derived in the previous sections the pre-computation of the voxel integral contributions is widely generalized for various voxel data. The time-consuming pre-computation only depends on the polynomial degree \(p\) and the number \((m_x \times m_y \times m_z)\) of sub-cells per finite cell used for integration. This effort corresponds to the results shown in column 3 of Table 1.
<table>
<thead>
<tr>
<th>p-deg.</th>
<th>pre-computation</th>
<th>sub-cell integration</th>
<th>accel. factor</th>
</tr>
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<td>38</td>
<td>271</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>4</td>
<td>5.09</td>
<td>5755</td>
<td>1130</td>
</tr>
<tr>
<td>5</td>
<td>11.19</td>
<td>18430</td>
<td>1647</td>
</tr>
<tr>
<td>6</td>
<td>22.46</td>
<td>49577</td>
<td>2207</td>
</tr>
</tbody>
</table>

Table 1: Computation time for the pre-computation scheme and a standard FCM Gauss quadrature on $10^3$ sub-cells/finite cell.

### 4.1 Unit cube with heterogeneous material properties

The pre-computation concept of section 3.1 allows the use of various integration methods to solve the governing integral equations. In [37] the symbolic computation framework *GiNaC* [5] was used to exactly pre-compute the cell stiffness. In this contribution we basically focus on numerical integration by Gauss quadrature. It is worth to mention that due to the orthogonal cell structure of the Finite Cell Method, the polynomial character of the Ansatz functions is preserved in the governing integral equations, thus allowing exact integration with $(p+1)$ quadrature points for each coordinate direction. With the following benchmark problem, developed in [24], the accuracy of the method is demonstrated and it is further shown that due to the decomposition of the governing integrals into sub-integrals, reduced integration schemes are applicable without

Figure 3: Geometry and symmetry boundary conditions at faces $x = -0.5$, $y = -0.5$ and $z = 0$ of a unit cube model (left), discrete heterogeneous distribution of the Young’s modulus (right). Location $A$ marks the point of interest that was considered in the analysis.

loss of accuracy thus reducing the numerical effort significantly. A two-point Gauss quadrature and the Simpson rule are applied exemplarily to demonstrate the well-behaving performance for low order integration.

We consider a voxelized unit cube of heterogeneous material properties that was derived from a continuous model, generated with the relation $E = (x + 10)^2(y + 10)(z + 1) \text{[N/mm}^2\text{]}$ for the Young’s modulus and a constant Poisson ratio of $\nu = 0.3$. Geometry and boundary conditions are provided in Figure 3.

For the FCM approach the cube is discretized into $2 \times 2 \times 2$ cells where each cell contains $10 \times 10 \times 10$ voxel. For each voxel a Young’s modulus is computed from the above equation and the corresponding value is assumed to be constant within the voxel. Based on this FCM discretization a p-extension with $p = 1, 2, 3, \ldots, 8$ is performed applying the pre-computation scheme.

Diagram 4a reveals the modeling error of the applied voxel resolution with respect to the continuous model following the given material distribution function for $E$. The diagram shows the relative error in energy norm $e_r$ for a uniform p-refinement of the voxelized cube model and the continuous model

$$e_r = \sqrt{\frac{|U_{EX} - U_{FCM}|}{U_{EX}}} \times 100 \text{ [%]}$$

with $U_{EX/FCM}$ denoting the strain energy of the exact solution and the numerical approximation, respectively. The reference value $U_{EX}$ for both curves was extrapolated from the p-FEM solution for the polynomial degrees $p = 8, 9, 10$.

Diagram 4b shows the accuracy of the pre-computation scheme for various integration schemes with regard to the exactly integrated solution with the composed sub-cell integration scheme shown in [24, 10]. The result of the pre-computation scheme exactly meets the solution of the composed sub-cell integration, as it was expected from section 3.1. The numerical effort for the pre-integration of the cell stiffness was significantly reduced by a decrease of the number of quadrature points for the sub-integrals of (13). Two quadrature points in each local direction of the sub-cell proved to be sufficient for all applied polynomial degrees to achieve solutions without any noticeable error in energy norm. An analogous result was observed for the application of the Simpson rule. For both methods identical results were achieved for all polynomial degrees (Fig. 4b).

Figure 5 illustrates the convergence of the cube example in terms of the displacements $u_x$, $u_z$ (Diagram 5b (left)) and $u_y$ (Diagram 5b (right)) for reference point $A$ and the relative error of the displacements $u_y$ (Diagram 5a) for a uniform p-refinement. The diagrams show curves for all tested integration schemes with almost identical results. The only slight deviation of $< 0.4\%$ of the Simpson rule from the Gauss quadrature results is noticeable in diagram 5a for the polynomial degrees 7 and 8.

### 4.2 Trabecular bone biopsy model

In the following example a specimen of human trabecular bone is analyzed. The example was chosen to demonstrate the accuracy and efficiency of the pre-computation approach for a highly
complex geometry in comparison to the analysis with the classical h-FEM. A reference solution has been computed with the FCM and compared to a standard h-version FEM in [10]. The material properties and boundary conditions are provided in Figure 6.

For a comparison the reaction force of the upper surface of the bone model is computed. The

Figure 4: Error and convergence analysis of the voxel-cube benchmark.
original model shown in Figure 6 is provided as a triangulated surface given in STL format from BARUFFALDI AND PERILLI [3]. The Finite Element reference solution was obtained from an analysis with the commercial software ABAQUS [29] applying 10-node tetrahedral elements. Several tetrahedral meshes of various densities were generated with the mesh generator NETGEN [27]. Table 2 provides the time effort for mesh generation and analysis of all generated FE-models. The reference FE-solution converges to a vertical reaction force of approximately 

\[ -2470N. \]

For the analysis with the Finite Cell Method four voxel models of various resolutions were generated with an octree-based voxel decomposition [36]. The various resolutions and corresponding results obtained from the pre-computation scheme analysis are provided in Figure 7. The convergence curves of the reaction forces are obtained with a uniform p-refinement. The coarser models (A and B) applied \( 5 \times 5 \times 5 \) voxels per finite cell while the finer models (C and D) were analyzed with \( 10 \times 10 \times 10 \) voxels per finite cell. With convergence to a value of 

\[ -2430N \]

the reaction force of the models C and D stay below a relative error of 2%.

The implementation of the FCM with pre-computation scheme was parallelized for multi-core architectures applying the OpenMP library [7] to exploit full performance of the developed method. Figure 8 shows the parallel efficiency for the computation of the stiffness of model C for polynomial degrees up to order 6. The efficiency clearly increases with the size of the cell matrices and settles down at \( \sim 80\% \) and reaches a maximum of 87% at \( p = 3 \). The code parallelization includes also the assembly and the solution of the governing system of equations (not part of Fig. 8). Whereas the assembly effort remains nearly constant, the solver profits from the parallel approach by a factor of \( \sim 3 \).

<table>
<thead>
<tr>
<th>refinem.</th>
<th># Elements</th>
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<th>Analysis</th>
<th>Total</th>
</tr>
</thead>
<tbody>
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<td>3437623</td>
<td>12083</td>
<td>43431</td>
<td>55514</td>
</tr>
</tbody>
</table>

Table 2: Time effort for mesh generation (NETGEN) and h-FEM analysis (ABAQUS) of the human biopsy model, measured on a single core AMD Opteron 250 (2.4GHz)

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The STL format is frequently applied in rapid prototyping and it is used to interact with stereo-lithography machines

ABAQUS element type: C3D10, quadratic tetrahedron, full integration

The total computational time effort of the converged solution of $2430N$ at $p = 6$ (Fig. 7, curve C) was measured on a 8 core -Intel Xeon W5590, 3.33GHz work station with 145s. A comparable model size ($\sim 300000$ degrees of freedom) from the NETGEN/ABAQUS analysis showed a total computational time effort of $\sim 3300s$ (single core AMD Opteron 250, 2.4GHz) including mesh generation and analysis. The analysis result of $2482N$ for this model had not yet converged to the reference value of $2470N$ and required another four refinement steps until convergence to the final solution was observable with a total time effort of more than 15 hours.

4.3 Validated femur analysis

The following example simulates the elastic behavior of a human proximal femur under compression. With this example we demonstrate the potential of the method to accurately predict the in-vivo bone strength of a human femur. The numerical results are validated by an in-vitro experiment by YOSIBASH et al. [41]. In particular the example shows that the simulation of the qCT-scanned fresh frozen femur bone can be performed without a time-consuming segmentation.

Figure 9(a) shows the setup of the experiment. The femur is loaded with a $1000N$ compression load on the femur head and fully clamped at its distal face. Four uniaxial strain gauges are installed on the bone surface to measure surface strains at the femoral neck (SG1), the greater trochanter (SG2) and the lateral shaft (SG3, SG4). The vertical displacement is measured with a LVDT‡ installed at the plunger of the testing machine.

The numerical model was derived from a qCT-scan (Fig.9(b)) with a resolution of $512 \times 512 \times 200$ voxel and a spacing of $0.78125mm$ in the $x - y$-plane of the qCT-slices and $0.75mm$ in $z$-direction. The bone is embedded in 667 finite cells, each covering $10 \times 10 \times 10$ voxel (Fig.9(c)). The boundary conditions are applied according to the experiment. Heterogeneous isotropic material properties are derived from a linear conversion of the Hounsfield Unit distribution obtained from the qCT-scan in an equivalent mineral density, calibrated by a scanned phantom device.

$$\rho_{EQM} = 10^{-3}(0.6822 \, HU - 5.48) \, [g/cm^3]$$  \hspace{1cm} (21)

With (21) the Young’s modulus for each voxel of the cortical and trabecular bone regions were found according to the relation [15]

$$E = 10200 \, (1.22 \, \rho_{EQM} + 0.0523)^{2.01} \text{ if } HU \leq 500$$  \hspace{1cm} (22)

$$E = 5307 \, (1.22 \, \rho_{EQM} + 0.0523) + 469 \text{ if } HU > 500 \, [MPa]$$  \hspace{1cm} (23)

Previous studies confirm that isotropic material properties suffice to obtain accurate results for this type of simulation [2, 22, 24].

The FCM analysis was performed for a polynomial degree of $p = 4$. At this $p$-degree the analysis showed a relative error in energy norm of 8% (Figure 10). A reference value for the relative error estimate was found from extrapolation of the strain energy values at $p = 4, 5, 6$.

Figure 11 shows the von Mises stress distribution from a computation with $p = 4$ along a vertical and horizontal cut through the femur that reveals stress concentrations in the lower femoral neck and the medial shaft area.

5. SUMMARY AND CONCLUSIONS

Table 3: Comparison of predicted and measured results and relative errors with respect to the experiment for a polynomial degree $p = 4$.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Uz</th>
<th>SG1</th>
<th>SG2</th>
<th>SG3</th>
<th>SG4</th>
</tr>
</thead>
<tbody>
<tr>
<td>point-wise</td>
<td>-300</td>
<td>-1303</td>
<td>440</td>
<td>-1303</td>
<td>441</td>
</tr>
<tr>
<td>error [%]</td>
<td>1.67</td>
<td>4.14</td>
<td>83.18</td>
<td>10.13</td>
<td>0.91</td>
</tr>
<tr>
<td>surface-avg.</td>
<td>NA</td>
<td>-1246</td>
<td>765</td>
<td>-1154</td>
<td>440</td>
</tr>
<tr>
<td>error [%]</td>
<td>NA</td>
<td>4.37</td>
<td>73.86</td>
<td>11.43</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table 3 compares the numerical prediction and the in-vitro experiment for surface strains ($\mu\epsilon$) and the vertical displacement ($\mu m$). The comparison includes pointwise values in terms of principal strains and surface averaged results obtained from several point results in the close vicinity of the strain gauge location. The results show good agreement between the predicted and measured values of the experiment. Location 2 is identified as outlier result most probably due to insufficient reading of the strain gauge. Numerical results that were found by a $p$-FEM analysis in [40] confirm the result as outlier and indicate a high sensitivity of the near neighborhood for this location. Even with the outlier included a linear regression analysis provides a very good correlation of the results with a correlation factor $R^2 > 0.92$.

Measured on an Intel Xeon W5590, 3.33GHz work station with 8 cores, the total computational time for the femur model (computed with $p = 4$) was below 10 seconds. In [38] it is shown that for a slightly coarser cell model ($p = 4$) an accurate and reliable solution can be retrieved with update rates below one second thus satisfying the requirements for responsive user interactive simulations.

5. Summary and conclusions

This paper has introduced a pre-computation scheme for voxel-based data models as an extension to the Finite Cell Method. The method is particularly able to handle geometric highly complex structures and problems with multi-material interfaces such as biological hard-tissue at negligible effort for modeling and simulation. Due to a fictitious domain approach of the Finite Cell Method the structure is embedded in a geometric simple domain of computation that is created on a Cartesian grid with no need for explicit, time-consuming and often error-prone mesh generation. Instead, the method exploits the structural and topological properties of the Cartesian grid and the voxel data. It is shown that the effort for the numerical evaluation of the governing cell

$\dagger$linear variable displacement transducer

integrals can be significantly reduced by a generalized, pre-computed set of matrices and vectors independent of any physical properties, only depending on geometric aspects. With the pre-computed data the true physical properties as stiffness and volume load are approximated at high accuracy and reduced cost in terms of time complexity. A specialization of the pre-computation scheme to qCT-derived voxel data enables a formulation that is independent of machine-specific characteristics of the scanner and provides a powerful tool for the analysis and simulation of structures with heterogeneous material properties as this is the case for any soft and hard tissue from biomedical applications. This approach has particularly proven to produce sufficient update rates for user-interactive simulations in a computational steering environment for patient specific surgery preparation [38], even for higher polynomial degrees. The implementation of the pre-computation scheme is parallelized for multi-core architectures and shows a parallel efficiency close to 90% on eight cores.

Accuracy, reliability and numerical efficiency is demonstrated with several examples. Due to the applied hierarchical high-order Ansatz spaces of the Finite Cell Method a uniform refinement of the polynomial degree can be easily performed during runtime to reduce the numerical error and to obtain numerically reliable results. Validation data from a femur experiment have proven to well correlate with the numerical prediction of the elastic response of the loaded femur in terms of strains and displacements thus underlining the high quality of the converged simulation results. A comparison of a FCM analysis with the modeling and simulation procedure of a standard h-FEM software reveals a dramatic improvement in terms of model generation and analysis. The Finite Cell approach shows a high potential for time-critical problems of voxel-based data models to significantly increase the performance of the simulation pipeline.

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References


[18] P. Magne. Efficient 3d finite element analysis of dental restorative procedures using micro-


Figure 5: Performance study at point A for various integration schemes.
Figure 6: Substructure of a human trabecular biopsy model with homogeneous isotropic material properties. The bone fragment is loaded on top with a prescribed displacement $u_0$ and fully clamped on the bottom face ($z = 0$). The analysis provides the reaction force of the upper surface for comparison.

Figure 7: Convergence plots for the FCM pre-computation analysis scheme of the human bone biopsy model indicating the total number of voxel and corresponding number of embedding finite cells.

Figure 8: Parallel efficiency of the multi-core parallel pre-computation of the model stiffness on 8 cores for a uniform $p$-refinement.

Figure 9: In-vitro experiment setup (a), qCT-voxel model of the proximal femur (b), finite cell model (c)
Figure 10: Convergence of the FCM analysis of the proximal femur: Relative error in energy norm for a uniform $p$-refinement ($p = 1, \ldots, 6$)

Figure 11: Voxelplot of the von Mises stress distribution of the proximal femur loaded on top with 1000N